ON THE GR-SPEEDS OF LIGHT AND PARTICLES (*A REMEMBRANCE OF THINGS PAST*)

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ABSTRACT. There are no limits for the speeds of light and particles in general relativity (GR). Four examples illustrate this basic result, which is too often neglected.

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1. – The recent interesting experiments by the OPERA Collaboration concerning the neutrino velocity [1] have given rise indirectly to renewed discussions about the velocities of light and particles in general relativity. – In this Note we emphasize the following fundamental concepts: *i*) In GR the light velocity can have *any* value between zero and infinite, according to the values of the metric tensor; *ii*) *Idem* for the particle velocity, which however must always be lower than the light velocity in the *same* pseudo-Riemannian manifold, but it can be greater than c (the light velocity *in vacuo*, *i.e.* in the absence of any field) in some instances.

We shall illustrate these statements with four simple examples.

2. – A preliminary consideration. In an axiomatic treatment of GR – see, *e.g.*, Hilbert [2] – the theory is divided in two distinct parts. The first part is devoted to the properties of the Einsteinian field equations, without any indication of the Riemannian or pseudo-Riemannian character of the interested manifolds.

In the second part, according to the *Erfahrung*, Hilbert specifies the nature of the interval $\mathrm{d}s^2$, which can be time-like, light-like, or space-like. The spacetime manifolds of physical interest must have a well-defined pseudo-Riemannian character.

The points *i*) and *ii*) of sect.**1** are implicitly contained in this axiom, *i.e.* they are a logical consequence of it.

3. – Let us consider the Einsteinian stationary field which exists in a reference frame uniformly rotating [3], [4]. The interval $\mathrm{d}s^2$ is:

$$(1) \qquad \mathrm{d}s^2 = \left(c^2 - \Omega^2 r^2\right) \mathrm{d}t^2 - 2\,\Omega\, r^2\, \mathrm{d}\varphi\, \mathrm{d}t - \mathrm{d}\sigma^2 \quad , \quad (c^2 > \Omega^2 r^2) \quad ,$$

$$\mathrm{d}\sigma^2 \equiv \mathrm{d}r^2 + r^2\mathrm{d}\varphi^2 + \mathrm{d}z^2 \quad , \tag{1'}$$

where Ω is the angular speed of rotation, and r, φ, z are cylindrical space-coordinates. Light motions are characterized by $\mathrm{d}s^2 = 0$. If Ω is *small*, we can keep only the terms of *first* order in $\Omega\, r^2/c$. We get:

$$\mathrm{d}t = \frac{\mathrm{d}\sigma}{c} \pm \frac{2\Omega}{c^2}\left(\frac{1}{2}\,r^2\mathrm{d}\varphi\right) \quad , \tag{2}$$

where the double sign ($\pm$) corresponds to the two directions of the light-ray motion.

If we put $r = a_1$, the Earth's equatorial radius ($a_1 = 6,378,137\,\mathrm{m}$), and $\Omega = \omega_\mathrm{E}$, the terrestrial rotation speed ($\omega_\mathrm{E} = 7.2921151467 \times 10^{-5}\,\mathrm{rad/s}$), for a full turn (within an optical fibre) of a light-ray around the equator we have:

$$\Delta t = \frac{2\pi a_1}{c} \pm \frac{2\omega_\mathrm{E}}{c^2}\,\pi a_1^2 \quad ; \tag{3}$$

since $\pi a_1^2 = 1.27802 \times 10^{14}\,\mathrm{m}^2$, and $2\omega_\mathrm{E}/c^2 = 1.6227 \times 10^{-21}\,\mathrm{m}^{-2}\mathrm{s}$, the second term at the right-side of (3) – which gives a so-called Sagnac effect – is equal to 207.4 ns.

The light velocity w is:

$$w = \frac{2\pi a_1}{\Delta t} = c \mp \omega_\mathrm{E}\, a_1 \quad ; \quad (\omega_\mathrm{E}\, a_1 = 0.4651011\,\mathrm{km/s}) \quad ; \tag{4}$$

we see that if the motions of the light-ray and of the reference frame happen in opposite directions, the light velocity is *greater* than c.

It is useful to compute w as an instantaneous speed. From $\mathrm{d}s^2 = 0$, we obtain:

$$\frac{\mathrm{d}t}{\mathrm{d}\varphi} = \pm\frac{\omega_\mathrm{E}\, a_1^2}{c^2} + \left(\frac{\omega_\mathrm{E}^2\, a_1^4}{c^4} + \frac{a_1^2}{c^2}\right)^{1/2} \approx \pm\frac{\omega_\mathrm{E}\, a_1^2}{c^2} + \frac{a_1}{c} \quad ; \tag{5}$$

from which:

$$a_1\frac{\mathrm{d}\varphi}{\mathrm{d}t} \approx \frac{c^2}{c \pm \omega_\mathrm{E}\, a_1} \quad ; \tag{6}$$

eq. (6) coincides with eq. (4) if we neglect $\omega_\mathrm{E}^2\, a_1^2$.

4. – With a good approximation, the Einsteinian metric near the Earth can be written as follows (see, *e.g.*, [4]):

$$\mathrm{d}s^2 \approx \left(1 + \frac{2V}{c^2}\right)c^2\mathrm{d}t^2 - 2\omega_\mathrm{E}\, r^2 \sin^2\vartheta\, \mathrm{d}\varphi\mathrm{d}t - \left(1 - \frac{2V}{c^2}\right)\mathrm{d}\sigma^2 \quad , \tag{7}$$

$$\mathrm{d}\sigma^2 \equiv \mathrm{d}r^2 + r^2\left(\mathrm{d}\vartheta^2 + \sin^2\vartheta\, \mathrm{d}\varphi^2\right) \quad , \tag{7'}$$

if we use isotropic polar coordinates r, ϑ, φ.

$$V \approx -\frac{GM_{\mathrm{E}}}{r}\left[1-\mathcal{J}_2\left(\frac{a_1}{r}\right)^2 P_2(\cos\vartheta)\right] \quad , \tag{8}$$

where: M_{E} is the Earth's mass, $\mathcal{J}_2$ is Earth's quadrupole moment coefficient, P_2 is the Legendre polynomial of degree 2; ($GM_{\mathrm{E}} = 3.986004418 \times 10^{14}\,\mathrm{m}^3\mathrm{s}^{-2}$; $\mathcal{J}_2 = 1.0826300 \times 10^{-3}$). From $\mathrm{d}s^2 = 0$, we have, keeping only the terms of *first* order in $\omega_{\mathrm{E}}\, r^2/c$ and V/c^2:

$$c\,\mathrm{d}t \approx \frac{\mathrm{d}\sigma \pm \omega_{\mathrm{E}}\, r^2 \sin^2\vartheta\, \mathrm{d}\varphi/c}{1+2V/c^2} \quad ; \tag{9}$$

for a full turn of a light-ray around the equator, we get [$P_2(\cos\pi/2) = -1/2$]:

$$a_1\frac{\mathrm{d}\varphi}{\mathrm{d}t} \approx \frac{c^2}{c \pm \omega_{\mathrm{E}}\, a_1}\left[1-\frac{2|V(a_1)|}{c^2}\right] \quad , \tag{10}$$

where

$$|V(a_1)| \approx \frac{GM_{\mathrm{E}}}{a_1}(1+0.54\times 10^{-3}) \approx \frac{GM_{\mathrm{E}}}{a_1} \quad , \tag{11}$$

from which:

$$\begin{aligned} a_1\frac{\mathrm{d}\varphi}{\mathrm{d}t} &\approx \frac{c^2}{c \pm \omega_{\mathrm{E}}\, a_1}\left(1-\frac{2GM_{\mathrm{E}}/c^2}{a_1}\right) \approx \\ &\approx (c \mp \omega_{\mathrm{E}}\, a_1) \times (1-1.4\times 10^{-9}) \approx \\ &\approx (c \mp 0.4651011)\,\mathrm{km/s} \quad . \end{aligned} \tag{12}$$

A result which is essentially coincident with that of the mere rotation.

5. – In Newton's theory, Laplace equation $\nabla^2 U = 0$ admits the two solutions $U = \pm GM/r$ for a material point. Analogous situation in GR. Let us start from the standard (Hilbert-Droste-Weyl) form of solution to Schwarzschild's problem of the gravitational field created by a point-mass M:

$$\mathrm{d}s^2 = \left(1-\frac{2m}{r}\right)c^2\mathrm{d}t^2 - \left(1-\frac{2m}{r}\right)^{-1}\mathrm{d}r^2 - r^2(\mathrm{d}\vartheta^2 + \sin^2\vartheta\,\mathrm{d}\varphi^2) \quad , \tag{13}$$

where $m \equiv GM/c^2$ is an integration constant, that is specified in this way because we wish to have the Newton solution at great values of r. Now, the Einstein equations $R_{jk} = 0$, $(j, k = 0, 1, 2, 3)$, admit also the following solution, which has a mere mathematical meaning (even if $\mathrm{d}s^2 \geq 0$):

$$\mathrm{d}s^2 = \left(1+\frac{k}{r}\right)c^2\mathrm{d}t^2 - \left(1+\frac{k}{r}\right)^{-1}\mathrm{d}r^2 - r^2(\mathrm{d}\vartheta^2 + \sin^2\vartheta\,\mathrm{d}\varphi^2) \quad , \tag{14}$$

where $k > 0$ is an integration constant; the condition $\mathrm{d}s^2 = 0$ implies:

$$\left|\frac{\mathrm{d}r}{\mathrm{d}t}\right|_{\text{LIGHT}} = c\left(1 + \frac{k}{r}\right) > c \quad ; \tag{15}$$

and for the radial geodesic of a test-particle we have:

$$\left|\frac{\mathrm{d}r}{\mathrm{d}t}\right|_{\text{PARTICLE}} = c\left(1 + \frac{k}{r}\right)\left[1 - |A|\left(1 + \frac{k}{r}\right)\right] \quad , \tag{16}$$

where A is an integration constant (for $A = 0$, we re-obtain eq. (15)). If $|A|$ is sufficiently small, *the particle velocity is greater than* c for all the values of r greater than a suitable value.

6. – The metric of Friedmann's model of universe corresponding to a positive space curvature is represented by the following interval $\mathrm{d}s^2$ (where x, y, z, r are dimensionless coordinates which give labels to the galaxies):

$$\mathrm{d}s^2 = c^2\mathrm{d}t^2 - A^2(r^2)\,F^2(t)\,\mathrm{d}\sigma^2 \quad ; \quad (r^2 = x^2 + y^2 + z^2) \quad , \tag{17}$$

$$\mathrm{d}\sigma^2 = \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2 \quad , \tag{17$'$}$$

$$A(r^2) = 1 + (r^2/4) \quad . \tag{17$''$}$$

The function $F(t)$ – *scale factor* – is determined by Einstein field equations (with a "dust-like" matter tensor), and is a usual cycloid. Thus, we have a cosmic periodic oscillation between $F = 0$ and $F = F_{\text{max}}$.

From $\mathrm{d}s^2 = 0$, we get:

$$\left|\frac{\mathrm{d}\sigma}{\mathrm{d}t}\right|_{\text{LIGHT}} = \frac{c}{A(r^2)\,F(t)} \quad ; \tag{18}$$

we see that the light velocity is infinite at every big-bang and at every big-crunch. –

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References

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